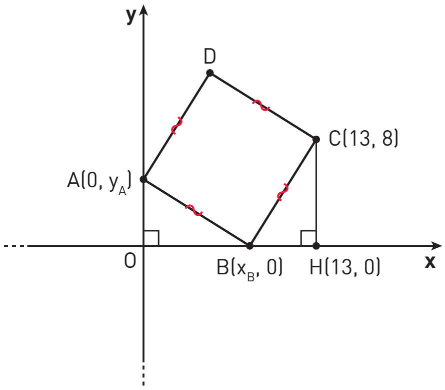
YOU & MATHS A square $ABCD$ is drawn with A on the y-axis, B on the x-axis, and C at the point $(13, 8)$. What is the area of the square?

(USA *Florida Atlantic University, Stuyvesant Alumni Math Competition*, 2000)

The problem gives us the following information about the vertices of the square:

- $A(0, y_A)$;
- $B(x_B, 0)$;
- $C(13, 8)$.

Let us sketch a graph of the square and let us also draw the projection H of point C onto the x-axis.



We can then notice that, for triangles OBA and BHC:

- they both are right-angled by hypothesis;
- $AB \cong BC$ by hypothesis;
- if we put $O\widehat{A}B = \alpha$, then $A\widehat{B}O = 90° - \alpha$ because the sum of the interior angles of a triangle is 180°. But since $A\widehat{B}O$, $C\widehat{B}A$ and $H\widehat{B}C$ are supplementary angles:

  $$H\widehat{B}C = 180° - A\widehat{B}O - C\widehat{B}A = 180° - (90° - \alpha) - 90° = \alpha.$$

  Then $OAB \cong HBC$.

Therefore, triangles OBA and BHC are congruent by the AAS (angle-angle-side) congruence criterion. This tells us that $OA \cong BH$ and that $OB \cong CH$. So, since:

$$\overline{OA} = |y_A - y_0| = |y_A|,$$

$$\overline{BH} = |x_H - x_B| = |x_C - x_B| = |13 - x_B|,$$

$$\overline{OB} = |x_B - x_0| = |x_B|,$$

$$\overline{CH} = |y_H - y_C| = |0 - 8| = 8,$$

we have that:

$$|y_A| = |13 - x_B| \text{ and } |x_B| = 8.$$

By the second equation, $x_B = \pm 8$. Then:

- $x_B = 8 \quad \to \quad |y_A| = |13 - 8| = 5 \quad \to \quad y_A = \pm 5$;
- $x_B = -8 \quad \to \quad |y_A| = |13 - (-8)| = 21 \quad \to \quad y_A = \pm 21$.

In order to understand which solution is the most plausible one, we need to find another condition to compare them with. We thus calculate the slopes of the line through AB and of the line through BC, which have to be perpendicular, so that $m_{AB} = -\dfrac{1}{m_{BC}}$.

Line through AB: $m_{AB}=\frac{y_B-y_A}{x_B-x_A}=\frac{0-y_A}{x_B-0}=-\frac{y_A}{x_B}$.

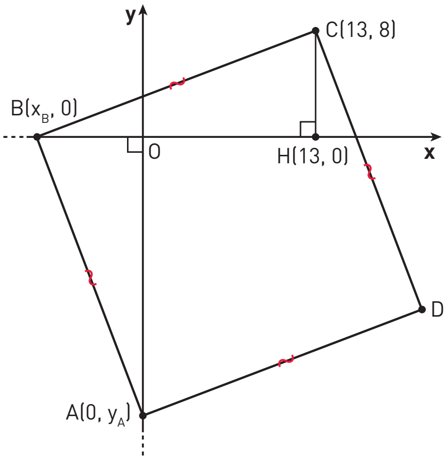
Line through BC: $m_{BC}=\frac{y_C-y_B}{x_C-x_B}=\frac{8}{13-x_B}$.

$$m_{AB}=-\frac{1}{m_{BC}} \quad\rightarrow\quad -\frac{y_A}{x_B}=-\frac{1}{\frac{8}{13-x_B}} \quad\rightarrow\quad \frac{y_A}{x_B}=\frac{13-x_B}{8} \quad\rightarrow\quad 8y_A=x_B(13-x_B).$$

This condition gives us additional information on x_B and y_A. Let us try and see if any of the solutions we have found earlier satisfy this condition:

- $x_B=8,\ y_A=5 \quad\rightarrow\quad 8\cdot 5=8(13-8) \quad\rightarrow\quad 40=40$ OK
- $x_B=8,\ y_A=-5 \quad\rightarrow\quad 8(-5)=8(13-8) \quad\rightarrow\quad -40=40$ impossible
- $x_B=-8,\ y_A=21 \quad\rightarrow\quad 8\cdot 21=(-8)(13-(-8)) \quad\rightarrow\quad 168=-168$ impossible
- $x_B=-8,\ y_A=-21 \quad\rightarrow\quad 8(-21)=(-8)(13-(-8)) \quad\rightarrow\quad -168=-168$ OK

We have thus proved that there are two possible solutions for x_B and y_A: we can either have $A(0, 5)$ and $B(8, 0)$ or $A(0, -21)$ and $B(-8, 0)$. Here is a sketch of how this second square looks like.



Let us finally use these figures to calculate the two possible lengths of the line segment AB and then the two possible areas of the square.

- If $x_B=8,\ y_A=5$:
  $\overline{AB}=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}=\sqrt{8^2+5^2}=\sqrt{64+25}=\sqrt{89}$;
  $\text{area}(ABCD)=\overline{AB}^2=89$.

- If $x_B=-8,\ y_A=-21$:
  $\overline{AB}=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}=\sqrt{8^2+(-21)^2}=\sqrt{64+441}=\sqrt{505}$;
  $\text{area}(ABCD)=\overline{AB}^2=505$.