YOU & MATHS Each of the following five statements is either true or false.

1. Statements 3 and 4 are both true.
2. Statements 4 and 5 are not both false.
3. Statement 1 is true.
4. Statement 3 is false.
5. Statements 1 and 3 are both false.

How many statements (1-5) are true?

A 0 B 1 C 2 D 3 E 4

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Let's first rewrite each statement using logical notation:

$1 \leftrightarrow 3 \wedge 4$

$2 \leftrightarrow \overline{\overline{4} \wedge \overline{5}}$

$3 \leftrightarrow 1$

$4 \leftrightarrow \overline{3}$

$5 \leftrightarrow \overline{1} \wedge \overline{3}$

We notice that the second implication can be simplified as follows:

$2 \leftrightarrow \overline{\overline{4} \wedge \overline{5}} = \overline{\overline{4}} \vee \overline{\overline{5}} = 4 \vee 5,$

using De Morgan's second law and the double negation property.

Since the second statement appears only in this proposition, let's start by studying this relationship with a truth table.

2	$4 \vee 5$	4	5
T	T	T	T
T	T	T	F
T	T	F	T
F	F	F	F

Now let's recall that $4 \leftrightarrow \overline{3}$ and $5 \leftrightarrow \overline{1} \wedge \overline{3}$, and update our table with this information.

2	$4 \vee 5$	4	$\overline{3}$	5	$\overline{1} \wedge \overline{3}$
T	T	T	T	T	T
T	T	T	T	F	F
T	T	F	F	T	T
F	F	F	F	F	F

We can also include the truth values of 3 and 1.

2	$4 \vee 5$	4	$\overline{3}$	3	5	$\overline{1} \wedge \overline{3}$	$\overline{1}$
T	T	T	T	F	T	T	T
T	T	T	T	F	F	F	F
T	T	F	F	T	T	T	?
F	F	F	F	T	F	F	T or F

Attention!

By doing so, we find out that the third line is inconsistent, as $\bar{1} \wedge \bar{3}$ cannot be true if $\bar{3}$ is false. Let's delete that line from our table, while adding an extra line for the two possible results of $\bar{1}$. We can also update the table with the values of 1.

2	4 ∨ 5	4	$\bar{3}$	3	5	$\bar{1} \wedge \bar{3}$	$\bar{1}$	1
T	T	T	T	F	T	T	T	F
T	T	T	T	F	F	F	F	T
F	F	F	F	T	F	F	T	F
F	F	F	F	T	F	F	F	T

So far, we have only used the information given by statements 2, 4 and 5. We would now like to use the information given by statement 3, that is, $3 \leftrightarrow 1$. Looking at our table, this relationship holds only twice, when both 3 and 1 have the same values, so let's delete the lines that are inconsistent with this new information.

2	4 ∨ 5	4	$\bar{3}$	3	5	$\bar{1} \wedge \bar{3}$	$\bar{1}$	1
T	T	T	T	F	T	T	T	F
F	F	F	F	T	F	F	F	T

We are finally ready to consider the information given by statement 1, that is, $1 \leftrightarrow 3 \wedge 4$. Let's first update the truth table by showing the values of $3 \wedge 4$.

2	4 ∨ 5	4	$\bar{3}$	3	3 ∧ 4	5	$\bar{1} \wedge \bar{3}$	$\bar{1}$	1
T	T	T	T	F	F	T	T	T	F
F	F	F	F	T	F	F	F	F	T

We notice that the relationship $1 \leftrightarrow 3 \wedge 4$ holds true only for the first line, while the second one is inconsistent. By deleting the second line, we get our final truth table:

2	4 ∨ 5	4	$\bar{3}$	3	3 ∧ 4	5	$\bar{1} \wedge \bar{3}$	$\bar{1}$	1
T	T	T	T	F	F	T	T	T	F

from which we notice that statement 1 is false, 2 is true, 3 is false, 4 is true and 5 is true. Since there are 3 true statements, our final answer is D.

It is possible to reach the same solution in a much shorter way. The line of reasoning goes as follows.
If 4 is true, then 3 is false. Hence, 3 and 4 cannot both be true. It follows that 1 cannot be true. Hence, both 1 and 3 must be false. We see then that 4 and 5 are true. It follows that 2 is true. The answer is D: 3 statements are true.