YOU & MATHS Suppose 1 and – 7 are roots of

$x^3 + ax^2 + bx + c = 0$, and $a + b = -15$.

Find the final root.

(USA *North Carolina State: High School Mathematics Contest,* 2002)

If 1 and -7 are roots of $x^3 + ax^2 + bx + c = 0$, then we can substitute them in the equation instead of x and the equality will still hold. It follows that:

- $1^3 + a \cdot 1^2 + b \cdot 1 + c = 0 \quad 1 + a + b + c = 0$
- $(-7)^3 + a \cdot -(-7)^2 + b \cdot (-7) + c = 0 - 343 + 49a - 7b + c = 0.$

We can put these equations in a linear system with the other condition on a and b and then solve for a, b and c by substitution.

Putting these values into the original equation of the problem, we obtain:

$x^3 + 4x^2 - 19x + 14 = 0.$

Let us solve it by using Ruffini's rule. We look for possible zeros of the polynomial $P(x) = x^3 + 4x^2 - 19x + 14$: they are fractions whose numerator is a divisor of the constant term and whose denominator is a divisor of the coefficient of x^3.

Integer divisors of 14	1	−1	2	−2	7	−7	14	−14
Integer divisors of 1	1	−1						

The possible zeros of $P(x)$ are:

$S = \{\pm 1, \pm 2, \pm 7, \pm 14\}.$

We now need to substitute them in $P(x)$ to see if they actually are zeros of the polynomial. Let us remember that we know already from the problem that 1 and -7 are roots of $P(x)$, so we don't need to try them out.

$P(-1) = (-1)^3 + 4(-1)^2 - 19(-1) + 14 = -1 + 4 + 19 + 14 = 36$ NO

$P(2) = 2^3 + 4 \cdot 2^2 - 19 \cdot 2 + 14 = 8 + 16 - 38 + 14 = 0$ YES

We have found the third root of our polynomial: it is 2.