YOU & MATHS Let a and b be distinct real numbers for which:

$$\frac{a}{b} + \frac{a+10b}{b+10a} = 2.$$

Find $\frac{a}{b}$.

A 0.6 **B** 0.7 **C** 0.8 **D** 0.9 **E** 1

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First of all, let us define the domain of the equation. The denominators of the two fractions must be different than zero:

$b \neq 0$ and $b + 10a \neq 0$.

Moreover, let us keep in mind that the problem states that a and b are distinct real number. Therefore:

$a \neq b$.

In order to find $\frac{a}{b}$, we start by dividing and multiplying by b both the numerator and the denominator of the second fraction, as follows:

$$\frac{a}{b} + \frac{a+10b}{b+10a} = 2 \;\rightarrow\; \frac{a}{b} + \frac{\left(\frac{a+10b}{b}\right)b}{\left(\frac{b+10a}{b}\right)b} = 2 \;\rightarrow\; \frac{a}{b} + \frac{\left(\frac{a}{b} + \frac{10\cancel{b}}{\cancel{b}}\right)b}{\left(\frac{\cancel{b}}{\cancel{b}} + \frac{10a}{b}\right)b} = 2 \;\rightarrow$$

$$\frac{a}{b} + \frac{\left(\frac{a}{b} + 10\right)\cancel{b}}{\left(1 + \frac{10a}{b}\right)\cancel{b}} = 2 \;\rightarrow\; \frac{a}{b} + \frac{\frac{a}{b} + 10}{1 + 10\frac{a}{b}} = 2.$$

(Note that we were able to simplify b because we assumed that $b \neq 0$.)

Let $\frac{a}{b} = x$. Then we can rewrite the equation above as:

$$x + \frac{x+10}{1+10x} = 2$$

and solve for x.

$$\frac{x(1+10x)+x+10}{1+10x} = \frac{2(1+10x)}{1+10x} \xrightarrow{\text{multiplying both sides by } 1+10x} x + 10x^2 + x + 10 = 2 + 20x \;\rightarrow\; 10x^2 - 18x + 8 = 0 \;\rightarrow$$

$$2(5x^2 - 9x + 4) = 0 \;\rightarrow\; 5x^2 - 9x + 4 = 0$$

We calculate the discriminant and the solutions of the equation.

$$\Delta = 81 - 4 \cdot 5 \cdot 4 = 81 - 80 = 1$$

$$x = \frac{9 \pm \sqrt{1}}{2 \cdot 5} = \frac{9 \pm 1}{10} = \begin{cases} \frac{10}{10} = 1, \\ \frac{8}{10} = \frac{4}{5}. \end{cases}$$

Recalling that $x = \frac{a}{b}$, we have thus proved that $\frac{a}{b} = \frac{4}{5} = 0.8$ or $\frac{a}{b} = 1$.

If we check the initial conditions ($b \neq 0$, $b + 10a \neq 0$ and $a \neq b$), we notice that 1 is not an acceptable solution as in that case it would hold $a = b$. Our final answer is therefore C.