YOU & MATHS Two circles C_1 and C_2 have a common chord GH. Point Q is chosen on C_1 so that it is outside C_2. Lines QG and QH are extended to cut C_2 at V and W, respectively. Show that, no matter where Q is chosen, the length of VW is constant.

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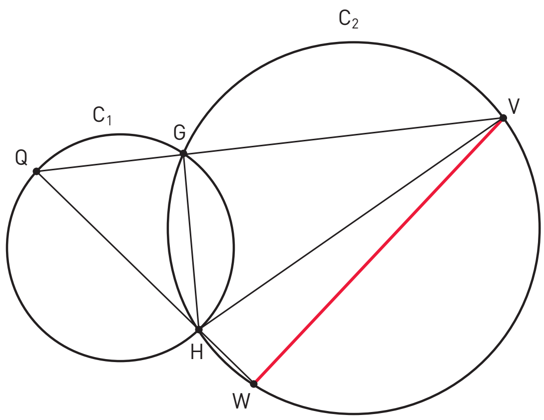
First of all, let us sketch a drawing of the two circles and the lines cutting them.

In order to prove that chord VW has constant length no matter where Q is, we have to prove that the angle inscribed in circle C_1 and subtended by VW is constant, that is, that angle $W\widehat{H}V$ always has the same measure.



From the problem, we know that the two circles are fixed and so is their common chord GH. Therefore, the angle inscribed in circle C_1 and subtended by GH, angle $H\widehat{Q}G$, will always have the same measure no matter where Q is. Similarly, the measure of angle $G\widehat{V}H$ is constant and independent from the position of Q.

We notice that

$$\begin{aligned} W\widehat{H}V &= 180° - V\widehat{H}Q \quad \text{because they are supplementary angles} \\ &= 180° - (180° - H\widehat{Q}G - G\widehat{V}H) \quad \text{as } V\widehat{H}Q,\ H\widehat{Q}G \text{ and } G\widehat{V}H \text{ are interior angles of triangle } QHV \\ &= H\widehat{Q}G + G\widehat{V}H. \end{aligned}$$

But $H\widehat{Q}G$ and $G\widehat{V}H$ have fixed measures as seen above, therefore angle $W\widehat{H}V$ is fixed as well and so is chord VW that subtends it.