YOU & MATHS Trevor's farm of mutant animals has 3-legged goats and 5-legged goats. In one pen he counts 83 legs and 23 heads; how many 5-legged goats are there?

(USA *Rice University Mathematics Tournament*, 2005)

Let us first identify the two equations hidden in our problem.

If we call x the number of 3-legged goats and y the number of 5-legged goats, then:

- since 83 is the total number of legs: $3x + 5y = 83$;
- since 23 is the total number of heads: $x + y = 23$.

We want to find the solution to the following system of linear equations:

$$\begin{cases} 3x + 5y = 83 \\ x + y = 23 \end{cases},$$

which we are going to solve using Cramer's rule. The determinant D of the matrix formed by the coefficients of x and y is:

$$D = \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = 3 \cdot 1 - 1 \cdot 5 = -2$$

and since $D \neq 0$, we know that the system is exactly determined and consistent.

To find the values of x and y, we calculate D_x by taking the above matrix and substituting the first column of coefficients of x with the constant terms of the two equations:

$$D_x = \begin{vmatrix} 83 & 5 \\ 23 & 1 \end{vmatrix} = 83 \cdot 1 - 23 \cdot 5 = 83 - 115 = -32.$$

We then calculate D_y starting once again from D and substituting the second column of coefficients of y with the constant terms of the two equations:

$$D_y = \begin{vmatrix} 3 & 83 \\ 1 & 23 \end{vmatrix} = 3 \cdot 23 - 1 \cdot 83 = 69 - 83 = -14.$$

We are now able to find x and y:

$$x = \frac{D_x}{D} = \frac{-32}{-2} = 16;$$

$$y = \frac{D_y}{D} = \frac{-14}{-7} = 7.$$

The number of 5-legged goats in the pen is 7.