YOU & MATHS Let m be a constant. The graphs of the lines $y = x - 2$ and $y = mx + 3$ intersect at a point whose x-coordinate and y-coordinate are both positive if and only if:

A $m = 1$.

B $m < 1$.

C $m > -\frac{3}{2}$.

D $-\frac{3}{2} < m < 0$.

E $-\frac{3}{2} < m < 1$.

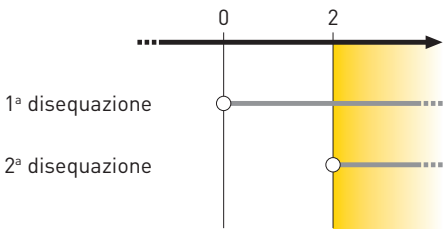
(USA *North Carolina State High School Mathematics Contest*, 2003)

First of all, let us remember that, in order for lines $y = x - 2$ and $y = mx + 3$ to intersect, they must not be parallel. That is, their slope must not be the same. Therefore $m \neq 1$.

The point of intersection of the two lines has both coordinates $(x, x - 2)$ and $(x, mx + 3)$, as it belongs to both lines. The problem gives us the condition that these coordinates must be positive, so we set up a linear system for the first line:

$$\begin{cases} x > 0 \\ x - 2 > 0 \end{cases} \rightarrow \begin{cases} x > 0 \\ x > 2 \end{cases}.$$

As we can see from the sketch below, the solution is $x > 2$.



Since the point belongs to both lines, we also have the condition on the equality of the y-coordinates:

$$mx + 3 = x - 2 \rightarrow mx - x = -2 - 3 \rightarrow (m - 1)x = -5.$$

For what was said at the beginning, we know that $m \neq 1$. We can then divide both sides of the equation by $(m - 1)$:

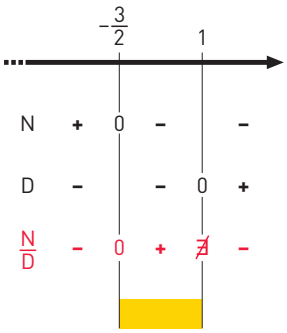
$$x = \frac{-5}{m - 1}.$$

We found out, by looking at the linear system above, that $x > 2$. Therefore we have:

$$\frac{-5}{m-1} > 2 \rightarrow \frac{-5}{m-1} - 2 > 0 \rightarrow \frac{-5 - 2(m-1)}{m-1} > 0 \rightarrow \frac{-5 - 2m + 2}{m-1} > 0 \rightarrow \frac{-2m - 3}{m-1} > 0.$$

This is a rational inequality, which we solve in the usual way, by studying separately the signs of the numerator and denominator.

- $-2m - 3 > 0 \rightarrow 2m + 3 < 0 \rightarrow m < -\frac{3}{2}$
- $m - 1 > 0 \rightarrow m > 1$



We can finally say that the values of m – such that the point of intersection of the two lines is in the first quadrant – are $-\frac{3}{2} < m < 1$. Our final answer is E.

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