YOU & MATHS Find all real numbers a such that the inequality $ax^2 - 2x + a < 0$ holds for all real numbers x.

A $a < -2$.

B $a < -1$.

C $a < 0$.

D $a < -2$ or $a > 2$.

E $a < -1$ or $a > 1$.

(USA *University of South Carolina: High School Math Contest*, 2002)

From what we have learnt so far, we know that the only case when a quadratic inequality holds for all real numbers x is when, for $a > 0$, $ax^2 + bx + c > 0$ and $\Delta < 0$.

In this problem, the inequality we are considering has a "less than" sign. Therefore, if we want its solution to be $\forall\, x \in \mathbb{R}$, we will have to change its sign by multiplying both sides of it by -1, as follows:

$$ax^2 - 2x + a < 0 \quad \rightarrow \quad -ax^2 + 2x - a > 0.$$

We now need to calculate its discriminant and set it less than zero, remembering that the coefficient of x^2 has to be positive (that is, $-a > 0$, which is equivalent to $a < 0$).

$$\frac{\Delta}{4} = \left(\frac{2}{2}\right)^2 - (-a)(-a) = 1 - a^2$$

$$\frac{\Delta}{4} < 0 \quad \rightarrow \quad 1 - a^2 < 0 \quad \rightarrow \quad a^2 > 1 \quad \rightarrow \quad a < -1 \vee a > 1.$$

As a has to be negative, the only acceptable solution is $a < -1$.