YOU & MATHS Given two similar triangles, one of which has twice the perimeter of the other, by what factor is the area of the larger triangle bigger than the smaller?

A 2 **C** $\sqrt{2}$ **E** None of these.

B 4 **D** $2\sqrt{2}$

(USA *North Carolina State High School Mathematics Contest,* 2004)

Let us call the smaller triangle ABC and the larger one $A'B'C'$. The perimeter of the smaller triangle is given by:

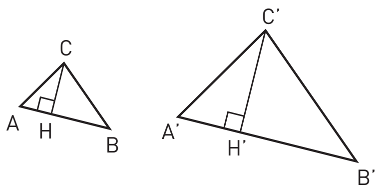
$$2p(ABC) = \overline{AB} + \overline{BC} + \overline{AC}.$$

It is given by the problem that the larger triangle has twice the perimeter of the smaller triangle, that is:

$$2p(A'B'C') = 2(\overline{AB} + \overline{BC} + \overline{AC})$$

We deduce that the sides of the larger triangle are twice as long as the corresponding sides of the smaller triangle.
Once we draw height CH in the smaller triangle, we have that:

$$\text{area}(ABC) = \frac{1}{2}\overline{AB} \cdot \overline{CH}.$$



In order to compare this expression with the area of $A'B'C'$, we first have to calculate the length of the corresponding height $C'H'$ in the larger triangle.
Let us consider the two triangles AHC and $A'H'C'$. They have:

- $A\widehat{H}C \cong A'\widehat{H'}C'$ as they are both right angles;
- $C\widehat{A}H \cong C'\widehat{A'}H'$ because ABC and $A'B'C'$ are similar triangles.

Therefore, by the AA similarity criterion, the two triangles are similar. In particular, as the corresponding sides AC and $A'C'$ have a proportionality factor of 2, the same thing must be true for sides CH and $C'H'$; in other words, $\overline{C'H'} = 2\overline{CH}$. We now have that:

$$\text{area}(A'B'C') = \frac{1}{2}\overline{A'B'} \cdot \overline{C'H'} = \frac{1}{2}2\overline{AB} \cdot 2\overline{CH} = 2\overline{AB} \cdot \overline{CH}$$

The ratio between the two areas is:

$$\frac{\text{area}(A'B'C')}{\text{area}(ABC)} = \frac{2\overline{AB} \cdot \overline{CH}}{\frac{1}{2}\overline{AB} \cdot \overline{CH}} = 4$$

and our final answer is B.